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Mechanical systems subjected to impulsive constraints

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Abstract. A geometrical description of mechanical systems subjected to impulsive non-holonomic constraints is given. Their motions are determined by means of suitable projection operators which allow us to evaluate the instantaneous jumps of momenta due to the action of impulsive reaction forces. Several paradigmatic examples are investigated from this viewpoint: a disk falling into a plane, a rolling ball hitting a wall and the collision of two balls.

1. Introduction

Let us start with three paradigmatic examples. Imagine a disk falling freely on a horizontal plane such that, after collision, it remains rolling on the plane. Assume that a ball rolls without slipping over a surface and, suddenly, it hits a wall. Or, finally, imagine the collision of two balls. The problem is how to describe in a geometrical way the dynamics of these mechanical systems. In classical, and also recent, books [1, 8, 9, 11] (see also [15]), these kinds of problems are widely treated, and some analytical solutions are given. The above examples are subjected to impulsive forces, that is, forces which act instantaneously, and have non-holonomic constraints. So, since the motion of a mechanical system described by a differential equation can be integrated if some initial data are given, such as positions and velocities, when an impulsive force acts, the new required initial data suffer a jump after the impulse. The problem then becomes how to know the new initial data after the impulse. This is accomplished mostly by using the particular physical conditions of the system: elasticity, Carnot's theorem, etc. The three examples above fit into the category of impulsive constraints, that is, the impulsive forces arise from the discontinuity of the constraints themselves. Indeed, they have one-sided holonomic constraints. In the first example, an impulsive non-holonomic constraint appears (the rolling condition) which remains after collision and in the second and third one, several possibilities appear depending on the conditions after the impulse.

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The aim of this paper is to provide a geometrical framework for describing systems subjected to impulsive constraints. Our inspiration is the geometrical description of non-holonomic mechanical systems as implicit differential equations introduced in [5] (see also [12, 13]). In our case, the constraint submanifold is a submanifold $\tilde{C}$ of the velocity space TQ with boundary, Q being the configuration manifold. In the interior of $\tilde{C}$ the permanent non-holonomic constraints act while on the boundary impulsive constraints also appear. We introduce the Chetaev bundle of reaction forces, and if two conditions of admissibility and compatibility are satisfied, the dynamics of the system is well defined. Moreover, a projector is defined which allows us to obtain the initial conditions after collision from those before collision. Projectors were previously used only to describe permanent constraints [10, 7, 2, 3]. As a by-product we obtain a geometric formulation of Carnot's theorem. Our results extend those previously obtained by Lacomba and Tulczyjew [6] for the case of one-sided holonomic constraints.

In section 11, the theory is extended to a more general situation, in the sense that the impulsive forces appear in a submanifold whose codimension is not necessarily 1. The geometrical description is very similar, and, assuming again the admissibility and compatibility conditions, the dynamics is ellucidated.

2. A mechanical motivation: Impulsive forces

We begin with a discussion of classical mechanical systems with impulsive forces, see [1, 8, 9, 11].

Consider a system of m particles in $\mathbb{R}^3$ such that the particle j has mass M_j. We introduce coordinates $(q^{3j-2}, q^{3j-1}, q^{3j})$ for the particle j. Suppose that $F_j = (F^{3j-2}, F^{3j-1}, F^{3j})$ is the force acting on the particle j.

The change of velocity of the particle j in an interval $[t_0, t_1]$ is determined by the system of integral equations

$$\dot{q}^\kappa(t_1) = \frac{\mathrm{d}q^\kappa}{\mathrm{d}t} = \frac{1}{M_j}\int_{t_0}^{t_1} F^\kappa(q, \dot{q}, \tau)\,\mathrm{d}\tau + \dot{q}^\kappa(t_0) \tag{1}$$

where $3j-2 \leqslant \kappa \leqslant 3j$. The integrals $\displaystyle\int_{t_0}^{t_1} F^\kappa\,\mathrm{d}\tau$, $3j-2 \leqslant \kappa \leqslant 3j$ are the components of the *impulse* of the force F_j and equation (1) establishes the relation between the impulse and the momentum change, i.e. 'impulse is equal to momentum change'. Equation (1) is a generalization of the Newton second law, in the sense that it allows us to consider the case of velocities with finite jump discontinuities (see [11]). This is precisely the case of *impulsive forces*, that is, an impulsive force F generates a finite non-zero impulse at some time instants. Then, if F is impulsive there exists an instant t_0 such that

$$\lim_{t\to t_0^+}\int_{t_0}^{t} F\,\mathrm{d}\tau = P \neq 0. \tag{2}$$

Equation (2) implies that the impulsive force has an infinite magnitude, i.e. $|F| = +\infty$, but we are assuming that its impulse P is well defined and finite. It can be mathematically thought of as a Dirac delta function concentrated at t_0.

Hereafter, we rename the coordinates and the forces as (q^κ) and (F^κ), $1 \leqslant \kappa \leqslant 3m = n$.

The impulsive forces may be caused by constraints. These kinds of constraints are called *impulsive constraints*. If we are in the presence of non-holonomic constraints of type $\Psi = 0$ where $\Psi = b_\kappa(q)\dot{q}^\kappa$, the constraint force is given by

$$F_\kappa = \mu b_\kappa$$

where μ is a Lagrange multiplier (see below). Then, this constraint is impulsive at t_0 if

$$\lim_{t \to t_0^+} \int_{t_0}^{t} \mu b_\kappa \, \mathrm{d}\tau = P_\kappa \neq 0.$$

The impulsive force may be caused by different circumstances: the function b_κ is discontinuous at t_0, the Lagrange multiplier μ is discontinuous at t_0 or both. We restrict ourselves to the case of smooth constraints Ψ, so that the impulsive force is caused by a discontinuity of the Lagrange multiplier.

Now, we derive the equations for impulsive motion (see [1]) by using d'Alembert's principle of constrained motion.

Let L be a Lagrangian system subjected to non-holonomic constraints $\Phi^A = 0$, where $\Phi^A = a^A_\kappa(q)\dot{q}^\kappa$, $1 \leqslant A \leqslant m$. The equations of motion of the system are derived from d'Alembert's principle:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} p_\kappa - \frac{\partial L}{\partial q^\kappa} - Q_\kappa \right) \delta q^\kappa = 0 \tag{3}$$

with δq^κ denoting the virtual displacements verifying the conditions

$$a^A_\kappa \delta q^\kappa = 0 \tag{4}$$

where Q_κ are non-conservative ordinary forces (non-impulsive) and $p_\kappa = \partial L / \partial \dot{q}^\kappa$ are the linear momenta. By applying the classical procedure of Lagrange multipliers we obtain the equations

$$\frac{\mathrm{d}}{\mathrm{d}t} p_\kappa - \frac{\partial L}{\partial q^\kappa} = Q_\kappa + \lambda_A a^A_\kappa.$$

Moreover, if we are in the presence of impulsive constraints, then the virtual displacements δq^κ must also be compatible with these constraints. In fact, let $\Psi^r = b^r_\kappa \dot{n}q^\kappa$ be an independent set of impulsive constraints; then the virtual displacements must verify equation (4) and also the supplementary conditions

$$b^r_\kappa \delta q^\kappa = 0. \tag{5}$$

Since the discontinuous velocity changes are produced by the action of the impulsive constraints, then

$$\lim_{t \to t_0^+} \int_{t_0}^{t} \left[\left(\frac{\mathrm{d}}{\mathrm{d}t} p_\kappa - \frac{\partial L}{\partial q^\kappa} - Q_\kappa \right) \delta q^\kappa \right] \mathrm{d}t = 0.$$

Note that $\partial L / \partial q^\kappa$ and Q_κ are bounded and the variations δq^κ are not functions of time. Then if we integrate the equations of motion and take limits we finally obtain the relationship between the pre-impulse and post-impulse states:

$$[(p_\kappa)_{t_0{}^+} - (p_\kappa)_{t_0}]\delta q^\kappa = 0.$$

In other words, the change of momentum Δp_κ satisfies the following relations

$$\Delta p_\kappa \delta q^\kappa = 0$$

or, in terms of Lagrange multipliers to take into account conditions (4) and (5),

$$\Delta p_\kappa = \bar{\lambda}_A a^A_\kappa + \bar{\mu}_r b^r_\kappa. \tag{6}$$

Moreover, observe that the velocity changes always verify the condition

$$a^A_\kappa \Delta \dot{q}^\kappa = 0$$

because the constraints Φ^A are permanent. Thus, if the Lagrangian is of the form $L = T - V$ where T is the kinetic energy of a Riemannian metric g on the configuration space, that is, $L = \frac{1}{2} g_{\kappa\nu}(q)\dot{q}^\kappa \dot{q}^\nu - V(q)$, then the momentum changes Δp_κ satisfy the relation

$$a_\chi^A \Delta p_\kappa g^{\kappa\chi} = 0. \tag{7}$$

From (6) and (7), we obtain that

$$\bar{\lambda}_A a_\kappa^A a_\chi^B g^{\kappa\chi} + \bar{\mu}_r b_\kappa^r a_\chi^B g^{\kappa\chi} = 0$$

which determines the Lagrange multipliers $\bar{\lambda}_A$ in terms of the Lagrange multipliers $\bar{\mu}_r$, since the matrix $(a_\kappa^A a_\chi^B g^{\kappa\chi})$ is regular because the matrix $(g^{\kappa\chi})$ is positive definite.

Then, in order to know the post-impulse momenta from the pre-impulse momenta, it is necessary to determine the remaining Lagrange multipliers, $\bar{\mu}_r$. For that, we need to use additional physical conditions about the considered mechanical system as freezing, elasticity, tangency conditions, etc.

3. Mechanical systems in implicit form

In this section, we consider the symplectic formulation of regular Lagrangian systems, and its implicit representations, see [13].

A mechanical system is given by a Lagrangian function L defined on the tangent bundle TQ of a configuration manifold Q. In what follows, we assume that L is regular, that is, the Hessian matrix $\left(\frac{\partial^2 L}{\partial \dot{q}^\kappa \partial \dot{q}^\chi}\right)$ is non-singular, where $(q^\kappa, \dot{q}^\kappa)$ are induced coordinates on TQ from local coordinates (q^κ) on Q. Denote by $\tau_Q : TQ \to Q$ the canonical projection.

We denote by J the canonical almost tangent structure, and by C the Liouville vector field on TQ locally defined by

$$J = \mathrm{d}q^\kappa \otimes \frac{\partial}{\partial \dot{q}^\kappa} \qquad C = \dot{q}^\kappa \frac{\partial}{\partial \dot{q}^\kappa}$$

respectively. Put $\omega_L = -\mathrm{d}(J^*(\mathrm{d}L))$, where J^* is the adjoint operator of J defined by $J^*\alpha(X) = \alpha(JX)$ for any one-form α and vector field X. The two-form ω_L is symplectic if and only if L is regular. In such a case, if $E_L = CL - L$ denotes the energy function, there exists a unique vector field ξ_L, solution of the equation

$$\mathrm{i}_{\xi_L}\omega_L = \mathrm{d}E_L$$

which in addition is a SODE (that is, $J\xi_L = C$) and whose integral curves are projected into the solutions of the Euler–Lagrange equations [4].

Since L is regular, ω_L defines a Poisson bivector Λ_L by

$$\Lambda_L(\alpha, \beta) = \omega_L(\sharp_L(\alpha), \sharp_L(\beta)) \qquad \text{for all one-forms } \alpha, \beta \text{ on } TQ$$

where $\sharp_L = \flat_L^{-1}$ and $\flat_L(X) = \mathrm{i}_X\omega_L$, for any vector field X on TQ. Thus, if the mechanical system is constraint-free ($\tilde{C} = TQ$) then $\xi_L = \sharp_L(\mathrm{d}E_L)$ yields the dynamics.

A simple computation shows that the induced correspondence between vector fields and one-forms on TQ is given by

$$\flat_L\left(\frac{\partial}{\partial q^\kappa}\right) = \left(\frac{\partial p_\kappa}{\partial q^\chi} - \frac{\partial p_\chi}{\partial q^\kappa}\right) \mathrm{d}q^\chi + \frac{\partial p_\kappa}{\partial \dot{q}^\chi} \mathrm{d}\dot{q}^\chi$$
$$\flat_L\left(\frac{\partial}{\partial \dot{q}^\kappa}\right) = -\frac{\partial p_\kappa}{\partial \dot{q}^\chi} \mathrm{d}q^\chi$$

where $p_\kappa = \frac{\partial L}{\partial \dot{q}^\kappa}$. Therefore, $\sharp_L = \flat_L^{-1}$ is given by

$$\sharp_L(\mathrm{d}q^\kappa) = -W^{\kappa\chi}\frac{\partial}{\partial \dot{q}^\chi}$$
$$\sharp_L(\mathrm{d}\dot{q}^\kappa) = W^{\kappa\chi}\frac{\partial}{\partial q^\chi} + R_{\mu\nu}W^{\nu\chi}W^{\kappa\mu}\frac{\partial}{\partial \dot{q}^\chi}$$

where $(W^{\kappa\chi})$ denotes the inverse matrix of the matrix whose entries are $W_{\kappa\chi} = \frac{\partial p_\kappa}{\partial \dot{q}^\chi}$, and

$$R_{\kappa\chi} = \frac{\partial p_\kappa}{\partial q^\chi} - \frac{\partial p_\chi}{\partial q^\kappa}.$$

Thus, we have

$$\xi_L = \dot{q}^\kappa \frac{\partial}{\partial q^\kappa} + \left(-\dot{q}^\mu \frac{\partial^2 L}{\partial \dot{q}^\nu \partial q^\mu} + \frac{\partial L}{\partial q^\nu}\right) W^{\nu\kappa}\frac{\partial}{\partial \dot{q}^\kappa}.$$

An alternative way to write the equations of motion based on the geometry of tangent and cotangent bundles was proposed by Tulczyjew [12–14].

Starting with coordinates (q^κ) on Q, we introduce coordinates

$$\begin{array}{ll} (q^\kappa, \dot{q}^\kappa) & \text{on } TQ \\ (q^\kappa, p_\kappa) & \text{on } T^*Q \\ (q^\kappa, \dot{q}^\kappa, a_\kappa, b_\kappa) & \text{on } T^*TQ \\ (q^\kappa, p_\kappa, \dot{q}^\kappa, \dot{p}_\kappa) & \text{on } TT^*Q. \end{array}$$

Tulczyjew defined a canonical diffeomorphism $\alpha : TT^*Q \longrightarrow T^*TQ$ as follows

$$\alpha(q^\kappa, p_\kappa, \dot{q}^\kappa, \dot{p}_\kappa) = (q^\kappa, \dot{q}^\kappa, \dot{p}_\kappa, p_\kappa).$$

Consider now the submanifold $D = \alpha^{-1}(\mathrm{d}L)$. Thus we have

$$D = \left\{(q^\kappa, p_\kappa, \dot{q}^\kappa, \dot{p}_\kappa) \in TT^*Q \mid \dot{p}_\kappa = \frac{\partial L}{\partial q^\kappa}, p_\kappa = \frac{\partial L}{\partial \dot{q}^\kappa}\right\}$$

which states that the local equations defining D are just the Euler–Lagrange equations for L.

Remark 3.1. It should be noted that α is a symplectomorphism from the symplectic manifold $(TT^*Q, \dot{\omega}_Q)$ to (T^*TQ, ω_{TQ}), where $\dot{\omega}_Q$ is the complete or tangent lift of the canonical symplectic form ω_Q on T^*Q to TT^*Q. Moreover, D is a Lagrangian submanifold of $(TT^*Q, \dot{\omega}_Q)$, even if L is not regular.

4. Non-holonomic one-sided constraints

We now consider now a modification of the formulation of the previous section, to include the Chetaev forces due to the presence of non-holonomic constraints.

Let the configuration space of a mechanical system be a manifold Q. The Lagrangian function $L : TQ \longrightarrow \mathbb{R}$ is supposed to be regular, so that ω_L is a symplectic form on TQ which defines a Poisson tensor Λ_L. We assume that the system is subjected to non-holonomic one-sided constraints determined by a *submanifold* $\tilde{C}$ of TQ *with boundary*, where the boundary $\partial\tilde{C}$ is assumed to be orientable.

A submanifold N with boundary of a differentiable manifold M is understood as a subset N of M, locally defined by equations of the form $\Phi^A(x) = 0$, $\Psi(x) \geqslant 0$; so, N is a manifold with boundary in the usual sense. Then, the interior of N (denoted by Int N) is a submanifold of M and the boundary ∂N of N is a submanifold of M of codimension 1 with respect to N.

We denote by $T\tilde{C}$ the tangent bundle of $\tilde{C}$, defined as follows. If $x \in \tilde{C}$, then $T_x\tilde{C}$ denotes the tangent vectors $X \in T_x(TQ)$, such that $X(\Phi^A) = 0$, for any A. If x is an interior point, then $T_x\tilde{C}$ is the usual tangent space (Int $\tilde{C}$ is a submanifold of TQ). The annihilator $(T\tilde{C})^0$ of $T\tilde{C}$ is locally generated by $\{\mathrm{d}\Phi^A\}$, i.e.

$$(T\tilde{C})^0 = \mathrm{span}\{\mathrm{d}\Phi^A\}.$$

Recall that a one-form α on TQ is semibasic if it belongs to the image of J^*.

We consider a vector subbundle of $J^*(T^*(TQ))_{|\tilde{C}}$ as follows

$$(F_1)_x = \begin{cases} \mathrm{span}\{J^*(\mathrm{d}\Phi^A)(x)\} & \text{if } x \in \mathrm{Int}\,\tilde{C} \\ \mathrm{span}\{J^*(\mathrm{d}\Phi^A)(x), J^*(\mathrm{d}\Psi)(x), \bar{f}_a(x)\} & \text{if } x \in \partial\tilde{C} \end{cases}$$

where $\bar{f}_a$, $1 \leqslant a \leqslant U$ are semibasic one-forms.

The set F_1 is a vector bundle over $\tilde{C}$ in an extended sense since not all of the fibres have the same dimension. It represents reaction forces of the constraints (also known as the Chetaev bundle). The one-forms $\{J^*(\mathrm{d}\Phi^A)(x)\}$ generate reaction forces due to the permanent constraints. The one-form $J^*(\mathrm{d}\Psi)(x)$ is due to the one-sided constraint. The one-forms $\{\bar{f}_1, \ldots, \bar{f}_U\}$ represent instantaneous reaction forces, due to the persistence of some instantaneous constraints.

We may assume that there are in addition external forces acting on the system. These forces are introduced as another vector bundle F_2 over $\tilde{C}$ which is a vector subbundle of $J^*(T^*(TQ))_{|\tilde{C}}$, i.e. it contains only semibasic one-forms. We also require that F_1 and F_2 have a trivial intersection, that is, $F_1 \cap F_2 = 0$, and we consider the Whitney sum $F_1 \oplus F_2$.

We assume that the following two conditions hold

(i) (admissibility)

$$\dim(T\tilde{C})^0 = \dim J^*(T\tilde{C})^0$$

(ii) (compatibility)

$$J^*(T\tilde{C})^0 \cap ((T\tilde{C})^0)^\perp = 0$$

where the orthogonal complement $((T\tilde{C})^0)^\perp$ is defined with respect to the Poisson structure Λ_L given by the symplectic form ω_L.

Remark 4.1. This condition is trivially satisfied if there are no permanent constraints.

Let $\alpha : TT^*Q \longrightarrow T^*TQ$ be the Tulczyjew diffeomorphism. We define an application that extends trivially α;

$$\tilde{\alpha} : TT^*Q \times_{TQ} T^*TQ \longrightarrow T^*TQ \times_{TQ} T^*TQ$$

by

$$\tilde{\alpha}(w, r) = (\alpha(w), r).$$

The map $\tilde{\alpha}$ is again a diffeomorphism.

Remark 4.2. Note that $TT^*Q \times_{TQ} T^*TQ$ is meaningful since it was defined as the set of couples $(w, r) \in TT^*Q \times T^*TQ$ such that

$$T\pi_Q(w) = \pi_{TQ}(r)$$

where $\pi_M : T^*M \to M$ denotes the canonical projection.

Let us consider the set

$$D = \tilde{\alpha}^{-1}(\mathrm{d}L + (F_1 \oplus F_2), F_1 \cap ((T\tilde{C})^0)^{\perp}).$$

Starting with coordinates (q^κ) on Q, consider the induced coordinates in TQ, T^*Q, T^*TQ, TT^*Q and TTQ as in the precedent section. Then

$$\tilde{\alpha}(q^\kappa, p_\kappa, \dot{q}^\kappa, \dot{p}_\kappa; q^\kappa, \dot{q}^\kappa, a_\kappa, b_\kappa) = (q^\kappa, \dot{q}^\kappa, \dot{p}_\kappa, p_\kappa; q^\kappa, \dot{q}^\kappa, a_\kappa, b_\kappa).$$

If $(w, r) \in TT^*Q \times_{TQ} T^*TQ$, we can locally write

$$w = (q^\kappa, p_\kappa, \dot{q}^\kappa, \dot{p}_\kappa)$$
$$r = (q^\kappa, \dot{q}^\kappa, a_\kappa, b_\kappa)$$

so that $(w, r) \in D$ is equivalent to

$$(q^\kappa, \dot{q}^\kappa, \dot{p}_\kappa, p_\kappa) \in \mathrm{d}L + (F_1 \oplus F_2)$$

and

$$(q^\kappa, \dot{q}^\kappa, a_\kappa, b_\kappa) \in F_1 \cap ((T\tilde{C})^0)^{\perp}.$$

The elements of F_2 are locally written as $(q^\kappa, \dot{q}^\kappa, f_\kappa, 0)$, or equivalently $f_\kappa\,\mathrm{d}q^\kappa$, so that $(w, r) \in D$ is equivalent to the following

$$(q^\kappa, \dot{q}^\kappa, \dot{p}_\kappa, p_\kappa) = \begin{cases} \left(q^\kappa, \dot{q}^\kappa, \dfrac{\partial L}{\partial q^\kappa} + \lambda_A \dfrac{\partial \Phi^A}{\partial \dot{q}^\kappa} + f_\kappa, \dfrac{\partial L}{\partial \dot{q}^\kappa}\right) \\ \qquad \text{if } (q^\kappa, \dot{q}^\kappa) \in \operatorname{Int}\tilde{C} \\ \left(q^\kappa, \dot{q}^\kappa, \dfrac{\partial L}{\partial q^\kappa} + \lambda_A \dfrac{\partial \Phi^A}{\partial \dot{q}^\kappa} + \tilde{\mu}\dfrac{\partial \Psi}{\partial \dot{q}^\kappa} + f_\kappa + \tilde{\nu}^a(\bar{f}_a)_\kappa, \dfrac{\partial L}{\partial \dot{q}^\kappa}\right) \\ \qquad \text{if } (q^\kappa, \dot{q}^\kappa) \in \partial\tilde{C} \end{cases}$$

and

$$b_\kappa = 0.$$

We conclude that $(w, r) \in D$ is written as

$$p_\kappa = \frac{\partial L}{\partial \dot{q}^\kappa}$$
$$\dot{p}_\kappa = \frac{\partial L}{\partial q^\kappa} + \lambda_A \frac{\partial \Phi^A}{\partial \dot{q}^\kappa} + \mu \frac{\partial \Psi}{\partial \dot{q}^\kappa} + f_\kappa + \nu^a(\bar{f}_a)_\kappa$$

where we let $\mu = \tilde{\mu}$, $\nu^a = \tilde{\nu}^a$ on $\partial\tilde{C}$ and $\mu = \mu^a = 0$ on $\operatorname{Int}\tilde{C}$ to unify notations, and also

$$b_\kappa = 0 \qquad a_\kappa \text{ arbitrary.}$$

Hereafter we will denote a_κ by Δp_κ.

4.1. Interpretation of the admissibility condition

Since

$$(T\tilde{C})^0 = \operatorname{span}\{\mathrm{d}\Phi^A\} \qquad J^*(T\tilde{C})^0 = \operatorname{span}\{J^*(\mathrm{d}\Phi^A)\}$$

the equality $\dim(T\tilde{C})^0 = \dim J^*(T\tilde{C})^0$ means that the map

$$J^* : (T\tilde{C})^0 \longrightarrow J^*(T\tilde{C})^0$$

is an isomorphism at each point of $\tilde{C}$. Hence, $\{\frac{\partial \Phi^A}{\partial \dot{q}^\kappa}\mathrm{d}q^\kappa\}$ are linearly independent, i.e. the reaction forces are independent.

4.2. Interpretation of the compatibility condition

We have

$$((T\tilde{C})^0)^\perp = \{\alpha \in T^*(TQ) \mid \Lambda_L(\alpha, (T\tilde{C})^0) = 0\}.$$

Since $(T\tilde{C})^0 = \mathrm{span}\{\mathrm{d}\Phi^A\}$, this means that $\Lambda_L(\alpha, \mathrm{d}\Phi^A) = 0$ for any A. So, if α belongs to the image of J^*, then it is a semibasic one-form. In local coordinates we have $\alpha = \alpha_\kappa \mathrm{d}q^\kappa$, and $\alpha \in ((T\tilde{C})^0)^\perp$ is equivalent to saying that $\Lambda_L(\alpha, \mathrm{d}\Phi^A) = 0$ for any A. This is in turn equivalent to

$$\alpha_\kappa \frac{\partial \Phi^A}{\partial \dot{q}^\chi} W^{\kappa\chi} = 0 \qquad \text{for any } A. \tag{8}$$

So, the compatibility condition

$$J^*((T\tilde{C})^0) \cap ((T\tilde{C})^0)^\perp = 0$$

is locally equivalent to the condition that the following matrix is non-singular

$$\left(W^{\kappa\chi} \frac{\partial \Phi^A}{\partial \dot{q}^\kappa} \frac{\partial \Phi^B}{\partial \dot{q}^\chi} \right). \tag{9}$$

The other condition in the definition of D requires that $r \in F_1 \cap ((T\tilde{C})^0)^\perp$.

If we write $r = \Delta p_\kappa \mathrm{d}q^\kappa$, then from (8) we have that $r \in ((T\tilde{C})^0)^\perp$ is the same as requiring

$$\Delta p_\kappa \frac{\partial \Phi^A}{\partial \dot{q}^\chi} W^{\kappa\chi} = 0 \qquad \text{for any } A. \tag{10}$$

But $r \in F_1$ means

$$r = \Delta p_\kappa \mathrm{d}q^\kappa = \bar{\lambda}_A J^*(\mathrm{d}\Phi^A) + \bar{\mu} J^*(\mathrm{d}\Psi) + \bar{\nu}^a \bar{f}_a = \left(\bar{\lambda}_A \frac{\partial \Phi^A}{\partial \dot{q}^\kappa} + \bar{\mu} \frac{\partial \Psi}{\partial \dot{q}^\kappa} + \bar{\nu}^a (\bar{f}_a)_\kappa \right) \mathrm{d}q^\kappa$$

with $\bar{\mu} = \bar{\nu}^a = 0$ on $\mathrm{Int}\,\tilde{C}$. Hence

$$\Delta p_\kappa = \bar{\lambda}_A \frac{\partial \Phi^A}{\partial \dot{q}^\kappa} + \bar{\mu} \frac{\partial \Psi}{\partial \dot{q}^\kappa} + \bar{\nu}^a (\bar{f}_a)_\kappa$$

and replacing conditions (9) so that r belongs to the intersection, we obtain

$$\bar{\lambda}_A \frac{\partial \Phi^A}{\partial \dot{q}^\kappa} \frac{\partial \Phi^B}{\partial \dot{q}^\chi} W^{\kappa\chi} + \bar{\mu} \frac{\partial \Psi}{\partial \dot{q}^\kappa} \frac{\partial \Phi^B}{\partial \dot{q}^\chi} W^{\kappa\chi} + \bar{\nu}^a (\bar{f}_a)_\kappa W^{\kappa\chi} = 0. \tag{11}$$

This means that if $\bar{\mu}$ and $\bar{\nu}^a$ are given, we can compute $\bar{\lambda}_A$. In particular, if $\bar{\mu} = 0$ and $\bar{\nu}^a = 0$ we have $\bar{\lambda}_A = 0$ for any A (which is the case identically on $\mathrm{Int}\,\tilde{C}$).

Remark 4.3. If the Lagrangian is of the form $L = T - V$ where T is the kinetic energy of a Riemannian metric g on the configuration space, that is, $L = \frac{1}{2} g_{\kappa\nu}(q)\dot{q}^\kappa \dot{q}^\nu - V(q)$, then the compatibility condition is automatically satisfied since the matrix $(W_{\kappa\chi})$ is just the matrix $(g_{\kappa\nu})$.

Remark 4.4. We have supposed that the boundary of $\tilde{C}$ is connected, but it is clear that we can consider the case of a boundary consisting of two or more components. We only need to define the Chetaev bundle F_1 at every component (see example 8.1).

5. Motions

We will describe the motions for a mechanical system subjected to one-sided non-holonomic constraints as defined in section 4.

A *motion* is a curve in $T^*Q \times_Q T^*TQ$, i.e. a pair of curves (η, φ) where η is a curve in T^*Q and φ is a curve in T^*TQ such that $\pi_Q \circ \eta = \tau_Q \circ \pi_{TQ} \circ \varphi = \gamma$. We assume that the projection curve γ in Q is continuous and differentiable from above. The curves η and φ are not continuous in general, but posses lateral limits and are differentiable from above. The *jumping curve* $\Delta\eta$ is defined as follows

$$\Delta\eta(t) = \tau_Q^*\eta(t^+) - \varphi(t^+)$$

where

$$\tau_Q^* : T^*_{\gamma(t)}Q \longrightarrow T^*_{\pi_{TQ}(\varphi(t))}(TQ).$$

The curve φ is only auxiliary, and only the jumping curve is relevant.

The equation of motion is the condition that the image of the curve $(\dot\eta, \Delta\eta)$ is contained in D. Thus if we write

$$\begin{aligned}
&\eta(t) = (q^\kappa(t), p_\kappa(t))\\
&\dot\eta(t) = (q^\kappa(t), p_\kappa(t), \dot q^\kappa(t), \dot p_\kappa(t))\\
&\Delta\eta(t) = (q^\kappa(t), \dot q^\kappa(t), \Delta p_\kappa, 0)
\end{aligned}$$

the above condition is equivalent to the following equations

$$\begin{aligned}
&p_\kappa = \frac{\partial L}{\partial \dot q^\kappa}\\
&\dot p_\kappa = \frac{\partial L}{\partial q^\kappa} + \lambda_A \frac{\partial \Phi^A}{\partial \dot q^\kappa} + \mu \frac{\partial \Psi}{\partial \dot q^\kappa} + f_\kappa\\
&\Delta p_\kappa = \bar\lambda_A \frac{\partial \Phi^A}{\partial \dot q^\kappa} + \bar\mu \frac{\partial \Psi}{\partial \dot q^\kappa} + \bar\nu^a (\bar f_a)_\kappa
\end{aligned}$$

the Lagrange multipliers $\lambda_A, \mu, \bar\lambda_A, \bar\mu$ satisfying the conditions stated in section 4. From these local equations it is clear that η is a curve of momenta, and $\Delta\eta$ is a curve which at each point gives the jump in momenta produced by the impulsive forces.

6. Holonomic one-sided constraints

In this section, we modify the constructions of section 4 for holonomic one-sided constraints. This is not a particular case of the non-holonomic situation.

It shoud be noted that the projection of the submanifold $\tilde C$ is not necessarily the whole Q. This is just the case of holonomic one-sided constraints. In fact, let C be a submanifold with boundary in Q, that is, C is locally defined by equations of the form $\phi^A(q) = 0, \psi(q) \geqslant 0$. From C we obtain a suitable submanifold $\tilde C$ of TQ with boundary given by the local equations

$$\Phi^A = (\phi^A)^c = 0 \qquad \Psi = \psi^v \geqslant 0$$

where $f^c = \mathrm{d}_T f$ (resp. $f^v = \pi_Q^* \circ f$) denotes the complete (resp. vertical) lift to TQ of a function f on Q. These equations mean that $\tilde C$ is locally defined as follows

$$\Phi^A(q^\kappa, \dot q^\kappa) = \dot q^\kappa \frac{\partial \phi^A}{\partial q^\kappa} = 0 \tag{12}$$

$$\Psi(q^\kappa, \dot q^\kappa) = \psi(q^\kappa) \geqslant 0. \tag{13}$$

The Chetaev bundle becomes

$$(F_1)_x = \begin{cases} \operatorname{span}\{(J^*\mathrm{d}\Phi^A)(x)\} & \text{if } x \in \operatorname{Int}\tilde{C} \\ \operatorname{span}\{(J^*\mathrm{d}\Phi^A)(x),\, \tau_Q^*(\mathrm{d}\psi)(x),\, \bar{f}_a(x)\} & \text{if } x \in \partial\tilde{C}. \end{cases}$$

Equations (12) and (13) mean that we are defining $\tilde{C}$ to be the usual tangent bundle TC of C. Another possibility would be to take the tangent bundle consisting of the interior tangent vectors at the points of the boundary, and the ordinary tangent space at the points of the interior of C, but as we will see later, that is not the better choice.

So, $\tilde{C}$ is defined along C. A direct computation shows that our formalism is the natural extension of the one previously developed by Lacomba and Tulczyjew [6].

7. Mixed constraints

Another interesting situation occurs when we have mixed constraints, that is, the functions Φ^A are ordinary non-holonomic constraints but we have in addition a holonomic one-sided constraint $\psi(q) \geqslant 0$. We can again define a submanifold with boundary $\tilde{C}$ of TQ given by the following equations

$$\Phi^A = 0 \qquad \Psi = \psi^v \geqslant 0,$$

which is locally written as follows

$$\Phi^A(q^\kappa, \dot{\kappa}) = 0 \tag{14}$$

$$\Psi(q^\kappa, \dot{\kappa}) = \psi(q^\kappa) \geqslant 0 \tag{15}$$

and the Chetaev bundle is given by

$$(F_1)_x = \begin{cases} \operatorname{span}\{(J^*\mathrm{d}\Phi^A)(x)\} & \text{if } x \in \operatorname{Int}\tilde{C} \\ \operatorname{span}\{(J^*\mathrm{d}\Phi^A)(x),\, \tau_Q^*(\mathrm{d}\psi)(x),\, \bar{f}_a(x)\} & \text{if } x \in \partial\tilde{C}. \end{cases}$$

8. Examples

Example 8.1 (Holonomic permanent and impulsive constraints). We describe the motion of a particle in the segment $x + y = 1$, $x \geqslant 0$ and $y \geqslant 0$, that is, the particle collides with the walls $x = 0$ and $y = 0$. In order to study this system we introduce the regular Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2)$$

and the constraints

$$\phi = x + y - 1 = 0$$
$$\psi_1 = x \geqslant 0$$
$$\psi_2 = y \geqslant 0.$$

The changes of momenta are given by

$$\Delta p_x = \bar{\lambda} + \bar{\mu}_1$$
$$\Delta p_y = \bar{\lambda} + \bar{\mu}_2$$

where $\bar{\lambda} = 0$ if $0 < x < 1$; $\bar{\mu}_1 = 0$ if $x > 0$ and $\bar{\mu}_2 = 0$ if $y > 0$.

Thus, if $x = 0$, the changes of momenta are determined by the algebraic equations

$$\Delta p_x = \bar{\lambda} + \bar{\mu}_1$$
$$\Delta p_y = \bar{\lambda}$$

and, by

$$\Delta p_x = \bar{\lambda}$$
$$\Delta p_y = \bar{\lambda} + \bar{\mu}_2$$

if $y = 0$.

The compatibility condition determines the multiplier $\bar{\lambda}$ in function of the multipliers $\bar{\mu}_1$ or $\bar{\mu}_2$. Therefore, if $x = 0$,

$$(\Delta p_x, \Delta p_y) = \left(\frac{\bar{\mu}_1}{2}, -\frac{\bar{\mu}_1}{2}\right)$$

and

$$(\Delta p_x, \Delta p_y) = \left(-\frac{\bar{\mu}_2}{2}, \frac{\bar{\mu}_2}{2}\right)$$

if $y = 0$.

If we suppose that e is the coefficient of restitution of both walls, $x = 0$ and $y = 0$, then we will obtain that

$$(\dot{x}_1 - \dot{x}_0, \dot{y}_1 - \dot{y}_0) = ((1 + e)\dot{y}_0, -(1 + e)\dot{y}_0)$$

or, in other words,

$$\dot{x}_1 = e\dot{y}_0$$
$$\dot{y}_1 = -e\dot{y}_0.$$

Example 8.2 (Non-holonomic impulsive constraints). While moving in a vertical plane xOy, a circular disk of radius R and mass m hits a rough wall determined by the axis Ox. Assuming that the motion is planar, the system possesses three degrees of freedom: the coordinates x and y of the centre of the disk and θ the angle between a point P of the disk and the axis Oy.

The system is described by the Lagrangian function

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + k^2\dot{\theta}^2)$$

where mk^2 denotes the moment of inertia of the disk. In addition, there are two impulsive constraints along the line $y = R$:

$$\psi = y - R$$
$$\psi' = \dot{x} - R\dot{\theta}.$$

The Chetaev bundle is determined by

$$F_1 = \mathrm{span}\{\mathrm{d}y, \mathrm{d}x - R\,\mathrm{d}\theta\}$$

if $y = R$ and $F_1 = 0$ if $y > R$ (i.e. free motion).

The algebraic equations for the instantaneous changes of momenta are

$$\Delta p_x = m\dot{x}_1 - m\dot{x}_0 = \bar{\nu}$$
$$\Delta p_y = m\dot{y}_1 - m\dot{y}_0 = \bar{\mu}$$
$$\Delta p_\theta = mk^2\dot{\theta}_1 - mk^2\dot{\theta}_0 = -R\bar{\nu}.$$

Without additional information these are the equations for the changes of momenta, i.e. the relation between the velocities after and before the collision. In fact, we need two new equations in order to determine the final velocity of the disk.

For example, we study two particular cases.

(i) After the collision the disk rolls on the Ox axis. In such a case, we obtain the following relations between the components of the final velocity:

$$\dot{y}_1 = 0 \qquad \dot{x}_1 - R\dot{\theta}_1 = 0.$$

After straightforward algebraic computations we obtain that

$$\dot{x}_1 = \frac{R^2\dot{x}_0 + k^2R\dot{\theta}_0}{k^2 + R^2}.$$

Compare with [1].

(ii) Elastic collision without slipping. Since the collision is elastic then $\dot{y}_1 = -\dot{y}_0$, and the no slipping condition means that $\dot{x}_1 - R\dot{\theta}_1 = 0$. Thus, the final velocity is given by

$$(\dot{x}_1, \dot{y}_1, \dot{\theta}_1) = \left(\frac{R^2\dot{x}_0 + k^2R\dot{\theta}_0}{k^2 + R^2}, -\dot{y}_0, \frac{R^2\dot{x}_0 + k^2R^2\dot{\theta}_0}{k^2 + R^2}\right).$$

Example 8.3. A sphere of radius r and mass 1 rolls without sliding on a horizontal plane. At the instant t_0, the sphere hits a rough wall determined by the plane yOz. Determine the post-impact velocities (see [8]).

The system is described by:

(i) the regular Lagrangian function

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + k^2(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta))$$

(ii) the permanent constraints (the sphere rolls without slidding on the xOy plane):

$$\begin{aligned}
\phi_1 &= \dot{x} - r\dot{\theta}\sin\psi + r\dot{\varphi}\sin\theta\cos\psi = 0\\
\phi_2 &= \dot{y} + r\dot{\theta}\cos\psi + r\dot{\varphi}\sin\theta\sin\psi = 0\\
\phi_3 &= \dot{z} = 0
\end{aligned}$$

or, after algebraic manipulations,

$$\begin{aligned}
\phi_1' &= \dot{x}\cos\psi + \dot{y}\sin\psi + r\dot{\varphi}\sin\theta = 0\\
\phi_2' &= \dot{x}\sin\psi - \dot{y}\cos\psi - r\dot{\theta} = 0\\
\phi_3' &= \dot{z} = 0
\end{aligned}$$

(iii) the instantaneous constraints on $x = R$:

$$\begin{aligned}
\Psi_1 &= \dot{x} = 0\\
\Psi_2 &= \dot{y} - r\dot{\varphi}\cos\theta - r\dot{\psi} = 0\\
\Psi_3 &= \dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi = 0
\end{aligned}$$

but since $r\Psi_3 = \Psi_1 - \phi_1$, we consider only Ψ_1 and Ψ_2 as instantaneous constraints.

The Chetaev bundle F_1 is given by

$$F_1 = \begin{cases} \operatorname{span}\{\cos\psi\,\mathrm{d}x + \sin\psi\,\mathrm{d}y + r\sin\theta\,\mathrm{d}\varphi, \sin\psi\,\mathrm{d}x - \cos\psi\,\mathrm{d}y - r\,\mathrm{d}\theta, \mathrm{d}z\} \\ \qquad \text{if } x > R \\ \operatorname{span}\{\cos\psi\,\mathrm{d}x + \sin\psi\,\mathrm{d}y + r\sin\theta\,\mathrm{d}\varphi, \sin\psi\,\mathrm{d}x - \cos\psi\,\mathrm{d}y - r\,\mathrm{d}\theta, \mathrm{d}z, \mathrm{d}x, \\ \qquad \mathrm{d}y - r\cos\theta\,\mathrm{d}\varphi - r\,\mathrm{d}\psi\} \qquad \text{if } x = R. \end{cases}$$

The relation between the pre-impact and post-impact momenta is obtained from the equations:

$$\begin{aligned}
\Delta p_x &= \lambda_1\cos\psi + \lambda_2\sin\psi + \mu\\
\Delta p_y &= \lambda_1\sin\psi - \lambda_2\cos\psi + \nu
\end{aligned}$$

$$\Delta p_z = \lambda_3$$
$$\Delta p_\theta = -r\lambda_2$$
$$\Delta p_\varphi = r\lambda_1 \sin\theta - r\nu\cos\theta$$
$$\Delta p_\psi = -r\nu.$$

From the compatibility condition we determine the Lagrange multipliers λ_1, λ_2 and λ_3 in terms of the multipliers μ and ν by using equation (11). Therefore,

$$\Delta p_x = \frac{r^2}{k^2+r^2}\mu$$
$$\Delta p_y = \frac{r^2}{k^2+r^2}\nu$$
$$\Delta p_z = 0$$
$$\Delta p_\theta = \frac{rk^2\sin\psi}{k^2+r^2}\mu - \frac{rk^2\cos\psi}{k^2+r^2}\nu$$
$$\Delta p_\varphi = -r\sin\theta\left(\frac{k^2\cos\psi}{k^2+r^2}\mu + \frac{k^2\sin\psi}{k^2+r^2}\nu\right) - r\nu\cos\theta$$
$$\Delta p_\psi = -r\nu$$

or, in terms, of velocities,

$$\Delta\dot{x} = \dot{x}_1 - \dot{x}_0 = \frac{r^2}{k^2+r^2}\mu$$
$$\Delta\dot{y} = \dot{y}_1 - \dot{y}_0 = \frac{r^2}{k^2+r^2}\nu$$
$$\Delta\dot{z} = \dot{z}_1 - \dot{z}_0 = 0$$
$$\Delta\dot{\theta} = \dot{\theta}_1 - \dot{\theta}_0 = \frac{r\sin\psi}{k^2+r^2}\mu - \frac{r\cos\psi}{k^2+r^2}\nu$$
$$\Delta\dot{\varphi} = \dot{\varphi}_1 - \dot{\varphi}_0 = -\frac{r}{\sin\theta}\left(\frac{\cos\psi}{k^2+r^2}\mu + \frac{\sin\psi}{k^2+r^2}\nu\right)$$
$$\Delta\dot{\psi} = \dot{\psi}_1 - \dot{\psi}_0 = \frac{r\cos\theta}{\sin\theta}\left(\frac{\cos\psi}{k^2+r^2}\mu + \frac{\sin\psi}{k^2+r^2}\nu\right) - \frac{r}{k^2}\nu.$$

In order to obtain a complete description of the post-impact velocities it is necessary to require additional information about the system.

For example, assume that the sphere remains rolling without sliding on the plane $x = 0$ after the impact. Then, we have the following relations between the post-impact velocities:

$$\dot{x}_1 = 0$$
$$\dot{y}_1 - r\dot{\varphi}_1\cos\theta - r\dot{\psi}_1 = 0.$$

Both conditions determine the Lagrange multipliers μ and ν as functions of the pre-impact velocities, that is,

$$\mu = -\frac{\dot{x}_0(k^2+r^2)}{r^2}$$
$$\nu = -\frac{k^2(k^2+r^2)(\dot{y}_0 - r\dot{\varphi}_0\cos\theta - r\dot{\psi}_0)}{r^2(2k^2+r^2)}.$$

The description of the collision of two balls runs along the same lines. In fact it can be modelled as if each one of the balls was hitting a vertical wall along the tangent plane to both at the impact point. If the two balls have equal radius, the one-sided impulsive constraint is given by

$$\Psi = (x - X)^2 + (y - Y)^2 + (z - Z)^2 \geqslant 4r^2$$

where (x, y, z) and (X, Y, Z) are the coordinates of the centres of the corresponding spheres and Ψ defines a codimension one boundary given by the smooth quadric:

$$(x - X)^2 + (y - Y)^2 + (z - Z)^2 = 4r^2.$$

9. A projector for the permanent non-holonomic constraints

Here we construct a projector which is defined whenever the system is subjected to permanent (i.e. those present all the time) non-holonomic constraints. This projector gives an algebraic way of showing that Lagrange multipliers $\bar{\lambda}_A$ can be computed in terms of $\bar{\mu}$ and $\bar{\nu}^a$, according to (11).

From the compatibility and admissibility conditions we have the following splitting of $T^*(TQ)$:

$$T_x^*(TQ) = (J^*(T\tilde{C})^0)_x \oplus ((T\tilde{C})^0)_x^\perp \tag{16}$$

for all $x \in \tilde{C}$ (see [2]). Thus, each covector $\alpha \in T_x^*(TQ)$ splits in a unique way as $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 \in (J^*(T\tilde{C})^0)_x$ and $\alpha_2 \in ((T\tilde{C})^0)_x^\perp$. Then, we can construct two complementary projectors $\bar{\mathcal{Q}}$ and $\bar{\mathcal{P}}$ defined by $\bar{\mathcal{Q}}_x(\alpha) = \alpha_1$ and $\bar{\mathcal{P}}_x(\alpha) = \alpha_2$.

Now, notice that $r = \Delta p_\kappa \, \mathrm{d}q^\kappa \in ((T\tilde{C})^0)^\perp$, then $\bar{\mathcal{P}}(r) = r$. Thus,

$$\begin{aligned}
\Delta p_\kappa \, \mathrm{d}q^\kappa &= \bar{\mathcal{P}}(\Delta p_\kappa \, \mathrm{d}q^\kappa) \\
&= \bar{\mathcal{P}}\left[\bar{\lambda}_A \frac{\partial \Phi^A}{\partial \dot{q}^\kappa} + \bar{\mu} \frac{\partial \Psi}{\partial \dot{q}^\kappa} + \bar{\nu}^a (\bar{f}_a)_\kappa\right] \mathrm{d}q^\kappa \\
&= \bar{\mu}\bar{\mathcal{P}}\left(\frac{\partial \Psi}{\partial \dot{q}^\kappa} \, \mathrm{d}q^\kappa\right) + \bar{\mathcal{P}}(\bar{\nu}^a (\bar{f}_a)_\kappa \, \mathrm{d}q^\kappa).
\end{aligned}$$

Therefore, we obtain the expression

$$\Delta p_\kappa \, \mathrm{d}q^\kappa = \bar{\mu}\bar{\mathcal{P}}\left(\frac{\partial \Psi}{\partial \dot{q}^\kappa} \, \mathrm{d}q^\kappa\right) + \bar{\nu}^a \bar{\mathcal{P}}((\bar{f}_a)_\kappa \, \mathrm{d}q^\kappa) \tag{17}$$

where the Lagrange multipliers $\bar{\lambda}_A$ which are associated to the permanent non-holonomic constraints Φ^A do not appear explicitly.

Example 8.1 continued. The projector $\bar{\mathcal{Q}}$ is

$$\bar{\mathcal{Q}} = -\frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \otimes (\mathrm{d}x + \mathrm{d}y).$$

If $x = 0$

$$\begin{aligned}
\Delta p_1 \, \mathrm{d}x + \Delta p_2 \, \mathrm{d}y &= \bar{\mathcal{P}}(\bar{\mu}_1 \, \mathrm{d}x) \\
&= \bar{\mu}_1 \left(\mathrm{d}x - \tfrac{1}{2}(\mathrm{d}x + \mathrm{d}y)\right) \\
&= \bar{\mu}_1 \left(\tfrac{1}{2}\mathrm{d}x - \tfrac{1}{2}\,\mathrm{d}y\right).
\end{aligned}$$

Thus $\Delta p_1 = \frac{\bar{\mu}_1}{2}$ and $\Delta p_2 = -\frac{\bar{\mu}_1}{2}$, as the regularity condition establishes.

Also, if $y = 0$,

$$\begin{aligned}\Delta p_1 \mathrm{d}x + \Delta p_2\, \mathrm{d}y &= \bar{\mathcal{P}}(\bar{\mu}_2\, \mathrm{d}y) \\ &= \bar{\mu}_2 \left(\mathrm{d}y - \tfrac{1}{2}(\mathrm{d}x + \mathrm{d}y)\right) \\ &= \bar{\mu}_2 \left(-\tfrac{1}{2}\mathrm{d}x + \tfrac{1}{2}\, \mathrm{d}y\right).\end{aligned}$$

Thus $\Delta p_1 = -\frac{\bar{\mu}_2}{2}$ and $\Delta p_2 = \dfrac{\bar{\mu}_2}{2}$.

10. A projector for the permanent impulsive constraints

In the case where some of the impulsive constraints remain after the impulsive force action, we define here another projector, which allows us to compute the new momenta in terms of the old momenta.

Denote by $\tilde{F}_1$ the vector subbundle, along the points of $\operatorname{Int} \tilde{C}$, locally generated by the one-forms

$$\operatorname{span}\left\{\bar{\mathcal{P}}\left(\frac{\partial \Psi}{\partial \dot{q}^\kappa}\, \mathrm{d}q^\kappa), \bar{\mathcal{P}}((\bar{f}_a)_\kappa\, \mathrm{d}q^\kappa\right)\right\}.$$

Let us consider a splitting

$$T^*TQ = \tilde{F}_1 \oplus \tilde{S}$$

along $\operatorname{Int} \tilde{C}$, i.e. we have $T_x^*TQ = (\tilde{F}_1)_x \oplus \tilde{S}_x$, for any point $x \in \operatorname{Int} \tilde{C}$.

Denote by $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{P}}$ the complementary projectors:

$$\tilde{\mathcal{Q}} : T^*TQ \longrightarrow \tilde{F}_1$$
$$\tilde{\mathcal{P}} : T^*TQ \longrightarrow \tilde{S}.$$

From (17) we obtain that

$$\tilde{\mathcal{P}}(\Delta p_\kappa\, \mathrm{d}q^\kappa) = 0$$

or

$$\tilde{\mathcal{P}}((p_\kappa)_1\, \mathrm{d}q^\kappa) = \tilde{\mathcal{P}}((p_\kappa)_0\, \mathrm{d}q^\kappa)$$

where $(p_\kappa)_0$ and $(p_\kappa)_1$, $1 \leqslant \kappa \leqslant \dim Q$ are the momenta before and after the impulsive force acts, respectively.

Suppose $(p_\kappa)_1\, \mathrm{d}q^\kappa$ belongs to $\tilde{S}$; then $\tilde{\mathcal{P}}((p_\kappa)_1\, \mathrm{d}q^\kappa) = (p_\kappa)_1\, \mathrm{d}q^\kappa$ and

$$(p_\kappa)_1\, \mathrm{d}q^\kappa = \tilde{\mathcal{P}}((p_\kappa)_0\, \mathrm{d}q^\kappa).$$

Then, it is possible to determine by projection the momenta after the impulse, $(p_\kappa)_1$, from the initial momenta $(p_\kappa)_0$.

Assume that we are working with a family of impulsive constraints $\Psi^a = \alpha_\kappa^a(q)\dot{q}^\kappa = 0$. We also assume that there are no permanent constraints or they have been eliminated through the projector $\bar{\mathcal{P}}$. If the impulsive constraints remain after the impulse, it means that

$$\alpha_\kappa^a(\dot{q}^\kappa)_1 = 0 \tag{18}$$

where we denote by $(\dot{q}^\kappa)_1$ the velocity after and by $(\dot{q}^\kappa)_0$ the velocity before the impulse takes place. If the Lagrangian is of kinetic type

$$L = g_{\kappa\chi}\dot{q}^\kappa\dot{q}^\chi - V(q)$$

we have that relation (18) linear in velocities, is transformed into a relation linear in momenta

$$\alpha_\kappa^a g^{\kappa\chi}(p_\kappa)_1 = 0. \tag{19}$$

If we now choose

$$\tilde{F}_1 = \mathrm{span}\{J^*(\mathrm{d}\Psi^A)\}$$
$$\tilde{S} = \{\alpha \in T^*(TQ) | \alpha(X_{\Psi^a}) = 0\}$$

we obtain the direct sum decomposition $T^*TQ = \tilde{F}_1 \oplus \tilde{S}$, since $\tilde{F}_1 \cap \tilde{S} = 0$ because the matrix $\tilde{\mathcal{C}} = (\tilde{\mathcal{C}}^{ab}) = (\alpha^a_\kappa \alpha^b_\chi g^{\kappa\chi})$ is regular ($g_{\kappa\chi}$ are the components of a Riemannian metric).

The projector $\tilde{\mathcal{P}} : T^*TQ \to \tilde{S}$ has the local expression $\tilde{\mathcal{P}} = \mathrm{id} - \tilde{\mathcal{Q}}$, where

$$\tilde{\mathcal{Q}} = \tilde{\mathcal{C}}_{ab} X_{\Psi^a} \otimes J^*(\mathrm{d}\Psi^b)$$

and $\tilde{\mathcal{C}}_{ab}$ are the entries of the inverse matrix of $\tilde{\mathcal{C}}$. It satisfies

$$(p_\kappa)_1 \, \mathrm{d}q^\kappa = \tilde{\mathcal{P}}((p_\kappa)_0 \, \mathrm{d}q^\kappa). \tag{20}$$

Alternatively, we obtain

$$\Delta p_\kappa \, \mathrm{d}q^\kappa = -\tilde{\mathcal{Q}}((p_\kappa)_0 \, \mathrm{d}q^\kappa). \tag{21}$$

Finally, the relationship between momenta before and after the instantaneous impulse is given by

$$(p_\kappa)_1 = (p_\kappa)_0 - \tilde{\mathcal{C}}_{ab} \alpha^a_{\kappa'} g^{\kappa'\chi'} \alpha^b_\kappa (p_{\chi'})_0$$

or in terms of velocities

$$(\dot{q}^\kappa)_1 = (\dot{q}^\kappa)_0 - \tilde{\mathcal{C}}_{ab} \alpha^a_{\kappa'} g^{\kappa\chi'} \alpha^b_{\chi'} (\dot{q}^{\kappa'})_0.$$

Remark 10.1. The result is similar if we suppose that there exist permanent non-holonomic constraints ϕ^A, in addition to the impulsive constraints $\Psi^a = \alpha^a_\kappa \dot{q}^\kappa$. Then, we choose

$$\tilde{F}_1 = \mathrm{span}\{\bar{\mathcal{P}}(J^*(\mathrm{d}\Psi^a))\}$$
$$\tilde{S} = \{\alpha \in T^*(TQ) | \alpha(X_{\bar{\Psi}^a}) = 0\}$$

where $\bar{\Psi}^a = \bar{\mathcal{P}}(J^*(\mathrm{d}\Psi^a))(\Gamma)$ being Γ an arbitary SODE vector field and $\bar{\mathcal{P}}$ is the projector constructed from the decomposition (16). In coordinates, if $\bar{\mathcal{P}}(\frac{\partial \Psi^a}{\partial \dot{q}^\kappa} \, \mathrm{d}q^\kappa) = \beta^a_\kappa \, \mathrm{d}q^\kappa$ then $\bar{\Psi}^a = \beta^a_\kappa \dot{q}^\kappa$.

If we suppose that the impulsive constraints remain after the impulse, then $\alpha^a_\kappa (\dot{q}^\kappa)_1 = 0$. Since we must also have

$$0 = \beta^a_\kappa (\dot{q}^\kappa)_1 \tag{22}$$

we deduce as in the previous case that

$$(p_\kappa)_1 \, \mathrm{d}q^\kappa = \tilde{\mathcal{P}}((p_\kappa)_0 \, \mathrm{d}q^\kappa). \tag{23}$$

or

$$\Delta p_\kappa \, \mathrm{d}q^\kappa = -\tilde{\mathcal{Q}}((p_\kappa)_0 \, \mathrm{d}q^\kappa) \tag{24}$$

where $\tilde{\mathcal{P}}$ is the projection over $\tilde{S}$, and $\tilde{\mathcal{Q}}$ is projection over $\tilde{F}_1$.

In coordinates,

$$(p_\kappa)_1 = (p_\kappa)_0 - \tilde{\mathcal{C}}_{ab} \beta^a_{\kappa'} g^{\kappa'\chi'} \beta^b_\kappa (p_{\chi'})_0$$

or in terms of velocities

$$(\dot{q}^\kappa)_1 = (\dot{q}^\kappa)_0 - \tilde{\mathcal{C}}_{ab} \beta^a_{\kappa'} g^{\kappa\chi'} \beta^b_{\chi'} (\dot{q}^{\kappa'})_0$$

where $\tilde{\mathcal{C}}_{ab}$ is the inverse matrix of $(\beta^a_\kappa g^{\kappa\chi} \beta^b_\chi)$.

As a consequence of our construction of the projector $\tilde{\mathcal{P}}$, we obtain Carnot's theorem, relating the kinetic energy before and after the impulse, provided instantaneous constraints remain.

Theorem 10.2 (Carnot's theorem). If the instantaneous constraints remain, then the following relationship between momenta before and after the action of the impulsive force holds:

$$(p_\kappa)_1 g^{\kappa\chi}(p_\chi)_1 - (p_\kappa)_0 g^{\kappa\chi}(p_\chi)_0 = -((p_\kappa)_1 - (p_\kappa)_0)g^{\kappa\chi}((p_\chi)_1 - (p_\chi)_0).$$

Proof. Since the impulsive constraints remain after the impulse we deduce, because of (22),

$$\begin{aligned}(p_\kappa)_1 g^{\kappa\chi}(p_\chi)_0 &= (p_\kappa)_1 g^{\kappa\chi}(\tilde{\mathcal{C}}_{ab}\beta^a_{\kappa'}g^{\kappa'\chi'}\beta^b_\chi(p_{\chi'})_0 + (p_\chi)_1)\\ &= (\beta^b_\chi(\dot{q}^\chi)_1)\tilde{\mathcal{C}}_{ab}\beta^a_{\kappa'}(\dot{q}^{\kappa'})_0 - (p_\kappa)_1 g^{\kappa\chi}(p_\chi)_1\\ &= (p_\kappa)_1 g^{\kappa\chi}(p_\chi)_1.\end{aligned}$$

So, we have that

$$\begin{aligned}((p_\kappa)_1 - (p_\kappa)_0)g^{\kappa\chi}((p_\chi)_1 - (p_\chi)_0) &= (p_\kappa)_1 g^{\kappa\chi}(p_\chi)_1 + (p_\kappa)_0 g^{\kappa\chi}(p_\chi)_0 - 2(p_\kappa)_1 g^{\kappa\chi}(p_\chi)_0\\ &= (p_\kappa)_0 g^{\kappa\chi}(p_\chi)_0 - (p_\kappa)_1 g^{\kappa\chi}(p_\chi)_1.\end{aligned}$$

□

Remark 10.3. Observe that Carnot's theorem is evident, since it is nothing other than the Pythagoras theorem:

$$|x - \tilde{\mathcal{P}}(x)|^2 = |x|^2 - |\tilde{\mathcal{P}}(x)|^2$$

since the projector $\tilde{\mathcal{P}}$ is orthogonal.

Example 8.2 continued. Consider the case where the constraints remain after the collision, i.e.

$$\dot{y}_1 = 0 \qquad \dot{x}_1 - R\dot{\theta}_1 = 0.$$

Denote by $\Psi_1 = \dot{y}$ and $\Psi_2 = \dot{x} - R\dot{\theta}$. The projector $\tilde{\mathcal{Q}}$ is defined by

$$\tilde{\mathcal{Q}} = \sum_{1\leqslant a,b\leqslant 2}(\tilde{\mathcal{C}}_{ab}X_{\Psi^a}\otimes J^*(\mathrm{d}\Psi^b))$$

where $\mathcal{C}_{ab}$ is the ab entry of the matrix

$$\begin{pmatrix} m & 0 \\ 0 & \frac{k^2 m}{k^2+R^2}\end{pmatrix}$$

and

$$X_{\Psi_1} = \frac{1}{m}\frac{\partial}{\partial y} \qquad X_{\Psi_2} = \frac{1}{m}\frac{\partial}{\partial x} - \frac{R}{k^2 m}\frac{\partial}{\partial\theta}.$$

Then, we obtain the post-impact momenta from the pre-impact momenta by using the matrix expression of the projector $\tilde{P}$ as follows

$$\begin{pmatrix}(p_x)_1\\(p_y)_1\\(p_\theta)_1\end{pmatrix} = \begin{pmatrix}\frac{k^2}{k^2+R^2} & 0 & -\frac{R}{k^2+R^2}\\ 0 & 1 & 0\\ \frac{Rk^2}{k^2+R^2} & 0 & \frac{R^2}{k^2+R^2}\end{pmatrix}\begin{pmatrix}(p_x)_0\\(p_y)_0\\(p_\theta)_0\end{pmatrix}.$$

11. Another type of impulsive motion

It is possible to extend our above constructions to a different setting where it is no longer necessary to restrict our study to submanifolds with a boundary of TQ.

We consider the submanifold H (without a boundary) of TQ locally determined by the vanishing of the permanent non-holonomic constraints $\Phi^A = 0$. Now we introduce a generalized vector bundle space F_1 over H where not all of the fibres have, in principle, the same dimension, such that it verifies

$$J^*(TH)^0 \subseteq F_1 \subseteq J^*(T^*(TQ))$$

along the points of H. For the sake of simplicity, we will suppose that the subset of points $x \in H$ where $(J^*(TH)^0)_x \subsetneq (F_1)_x$ is a submanifold $\bar{H}$ of H.

Assuming the admissibility condition, $\dim(TH)^0 = \dim J^*(TH)^0$, and the compatibility condition, $J^*(TH)^0 \cap ((TH)^0)^\perp = 0$, we obtain that the dynamics of the impulsive motion is also described by the subset

$$D = \alpha^{-1}(\mathrm{d}L + (F_1 \oplus F_2), F_1 \cap ((TH)^0)^\perp)$$

of $TT^*Q \times_{TQ} T^*TQ$, where F_2 is a subbundle of $J^*(T^*TQ)_{|H}$ which introduces the ordinary external forces.

Remark 11.1. As in remark 4.1, the admissibility condition is trivially satisfied if there are no permanent constraints.

The equations of motion are

$$\dot{p}_\kappa - \frac{\partial L}{\partial q^\kappa} \in F_1 \oplus F_2.$$

Also, the impulsive jumps of momenta are given by

$$\Delta p_k \, \mathrm{d}q^\kappa \in F_1.$$

If we suppose that $(F_1)_x$, $x \in \bar{H}$, is determined by

$$\mathrm{span}\{J^*(\mathrm{d}\Phi^A), \bar{f}^a\}$$

where $\bar{f}^a$ are semibasic one-forms, then

$$\Delta p_k \, \mathrm{d}q^\kappa = \bar{\lambda}_A J^*(\mathrm{d}\phi^A) + \bar{\mu}_a f^a.$$

The compatibility conditions imply that the Lagrange multipliers $\bar{\lambda}_A$ can be explicitly determined from the remaining Lagrange multipliers $\bar{\mu}_a$.

Now, we can define a projector $\bar{\mathcal{P}}$ as in section 9, and we obtain

$$\Delta p_k \, \mathrm{d}q^\kappa = \bar{\mu}_a \bar{\mathcal{P}}(f^a).$$

Example 11.2 (Oblique central impact of two particles). Consider two small smooth spheres modelized as particles, P_1 and P_2 of mass m and M in collision with one another. We choose the x- and y-axis, respectively, along the line of impact and along the common tangent to the surfaces in contact. Take coordinates (x, y) and (X, Y) for both particles, respectively. The total kinetic energy of the system is given by

$$K = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2).$$

The Chetaev bundle of reaction forces is given by

$$F_1 = \begin{cases} \{0\} & \text{if } x \neq X \quad \text{or} \quad y \neq Y \\ \mathrm{span}\{\mathrm{d}x - \mathrm{d}X, \mathrm{d}y - \mathrm{d}Y\} & \text{if } x = X \quad \text{and} \quad y = Y \end{cases}$$

The changes of momenta during the collision are then

$$\Delta p_x = \mu_1 \qquad \Delta p_y = \mu_2 \qquad \Delta p_X = -\mu_1 \qquad \Delta p_Y = -\mu_2.$$

Observe that these conditions give us the conservation of the total momenta in the collision, that is, the vector sum of the momenta after the collision is equal to the vector sum of these quantities before the collision.

By additional assumptions it is possible to find the values of the Lagrange multipliers μ_1 and μ_2.

Assuming that there are no vertical impulsive forces acting during the impact, the vertical component of the momentum of each particle is unchanged, i.e. $(p_y)_1 = (p_y)_0$ and $(p_Y)_1 = (p_Y)_0$. Thus, $\mu_2 = 0$.

If we know the coefficient of restitution e ($0 \leqslant e \leqslant 1$), then we have, therefore, the following relation between the relative velocities after and before the impact

$$(\dot{X})_1 - (\dot{x})_1 = -e((\dot{X})_0 - (\dot{x})_0).$$

After some computations we obtain that

$$\mu_1 = \frac{mM}{m+M}(1+e)((\dot{X})_0 - (\dot{x})_0)$$

and

$$(\dot{x})_1 = \frac{M}{m+M}(\dot{X})_0 + \frac{m}{m+M}(\dot{x})_0 + \frac{Me}{m+M}((\dot{X})_0 - (\dot{x})_0)$$
$$(\dot{X})_1 = \frac{M}{m+M}(\dot{X})_0 + \frac{m}{m+M}(\dot{x})_0 - \frac{me}{m+M}((\dot{X})_0 - (\dot{x})_0).$$

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